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THE LOCALIZATION OF SPACES WITH RESPECT TO HOMOLOGY

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§1. INTRODUCTION

MY MAIN purpose is to show that each generalized homology theory h_* determines an $\mathbf{h}_*$*-localization functor*, $E: \mathrm{Ho} \to \mathrm{Ho}$ and $\eta: 1 \to E$, where Ho is the pointed homotopy category of *CW* complexes. This localization is characterized by the universal property that $\eta_X: X \to EX$ is the terminal h_*-homology equivalence going out of E, i.e.

(i) $\eta_X: X \to EX$ induces $h_*(X) \approx h_*(EX)$, and

(ii) for any map, $f: X \to Y \in \mathrm{Ho}$ inducing $h_*(X) \approx h_*(Y)$, there is a unique map $r: Y \to EX \in \mathrm{Ho}$ with $rf = \eta_X$.

The plausibility and desirability of such a functor E were shown by Adams [2]. To obtain an existence proof (§3), I will construct an appropriate localization functor on the category of simplicial sets and will show that it induces the desired $\mathbf{h}_*$-localization functor on Ho. The backbone of this proof is in an appendix (§10–§12), where I introduce a version of simplicial homotopy theory in which the h_*-equivalences play the role of weak homotopy equivalences. I show that this theory fits into Quillen's framework of homotopical algebra [10], [11].

Special cases of the $\mathbf{h}_*$-localization, $X \to EX$, are familiar. If X is simply connected (or nilpotent) and $h_* = H_*(\ ; Z[J^{-1}])$ where $Z[J^{-1}]$ denotes a subring of the rationals,then $X \to EX$ is the usual $Z[J^{-1}]$-localization with $\Pi_* EX \approx Z[J^{-1}] \otimes \Pi_* X$. This case was discovered by Barratt–Moore (*ca* 1957, unpublished) and has subsequently been discovered and/or studied by various others, e.g. [4], [5], [7], [9], [11], [14], [15]. If X is simply connected (or nilpotent) and $h_* = H_*(\ ; Z_p)$ with p prime, then $X \to EX$ is the p-completion [5, p. 186] with $\Pi_n EX \approx \mathrm{Ext}(Z_{p^\infty}\, \Pi_n X) \oplus \mathrm{Hom}(Z_{p^\infty}, \Pi_{n-1} X)$. If in addition X is of finite type, then $X \to EX$ is the p-profinite completion [12], [14] with $\Pi_* EX$ given by the p-profinite completion of $\Pi_* X$. In [5] we gave various other examples of $\mathbf{H}_*(\ ; R)$-localizations where $R = Z_p$ or $R = Z[J^{-1}]$, and we constructed an "R-completion" $X \to R_\infty X$ which coincides with the $\mathbf{H}_*(\ ; R)$-localization provided X is "R-good".

A major part of this paper is devoted to the study of $\mathbf{H}_*(\ ; R)$*-local spaces*, i.e. spaces $X \in \mathrm{Ho}$ satisfying the equivalent conditions:

(i) $X \simeq EY$ for some $Y \in \mathrm{Ho}$

(ii) $\eta_X: X \simeq EX$.

For $R = Z[J^{-1}]$ and $R = Z_p$, I characterize (§5) the $\mathbf{H}_*(\ ;R)$-local spaces in terms of their homotopy groups. To do this, I introduce (§5) the *HR-localization* for groups and the *HZ-localization* for Π-modules; and I prove that $X \in \mathrm{Ho}$ is $\mathbf{H}_*(\ ;R)$-local if and only if the groups $\Pi_* X$ are HR-local and the $\Pi_1 X$-modules $\Pi_n X$ are HZ-local for $n \geq 2$. There is a step by step procedure (9.4) for constructing $\mathbf{H}_*(\ ;R)$-localizations of CW complexes by attaching cells so as to localize homotopy groups.

This paper is organized as follows. §2 contains categorical preliminaries on localization, §3 contains the existence proof for $\mathbf{h}_*$-localizations, §4 contains a determination of the $\mathbf{h}_*$-localization for nilpotent spaces where h_* is any connective homology theory, §5 contains the algebraic characterization of $\mathbf{H}_*(\ ;R)$-local spaces, §6 contains constructions of homology equivalences, §7 concerns the HR-localization of groups, §8 concerns the HZ-localization of Π-modules, §9 contains proofs and a step by step construction of $\mathbf{H}_*(\ ;R)$-localizations. There is a crucial appendix (§10, §11 and §12) which introduces "simplicial homotopy theory with respect to h_*" and contains a key technical result (11.1) used repeatedly in this paper.

I am particularly indebted to Frank Adams, Emmanuel Dror, and Dan Kan for their ideas and encouragement.

§2. LOCALIZATIONS IN CATEGORIES

I will explain some categorical notions which will be used repeatedly in this paper. In particular, I will show how a class $\mathscr{W}$ of morphisms in a category $\mathscr{C}$ can determine a "$\mathscr{W}$-localization" functor $E : \mathscr{C} \to \mathscr{C}$. The reader should keep in mind the easy example where $\mathscr{C}$ is the category of abelian groups and $\mathscr{W}$ consists of all $M \to N \in \mathscr{C}$ with $Q \otimes M \to Q \otimes N$ an isomorphism. In this case $E(M) \approx Q \otimes M$.

2.1. *$\mathscr{W}$-Localizations.* Given a morphism class $\mathscr{W}$ in a category $\mathscr{C}$, an object $D \in \mathscr{C}$ is *$\mathscr{W}$-local* if each $w : X \to Y$ in $\mathscr{W}$ induces a bijection $\mathrm{Hom}(Y, D) \approx \mathrm{Hom}(X, D)$. A *$\mathscr{W}$-localization* of an object $A \in \mathscr{C}$ is a morphism $w : A \to D$ with D $\mathscr{W}$-local and $w \in \mathscr{W}$. Any two $\mathscr{W}$-localizations of $A \in \mathscr{C}$ are naturally equivalent; indeed, a $\mathscr{W}$-localization $w : A \to D$ satisfies each of the universal conditions:

(i) *w is initial among the morphisms $f : A \to X$ with X $\mathscr{W}$-local.*

(ii) *w is terminal among the morphisms $f : A \to X$ with $f \in \mathscr{W}$.*

Moreover, if A is $\mathscr{W}$-local and $w : A \to D$ is a $\mathscr{W}$-localization, then w is an equivalence.

2.2. *$\mathscr{W}$-Localization functors.* Suppose each object of $\mathscr{C}$ has a $\mathscr{W}$-localization. Then there exist a functor and a transformation

$$E : \mathscr{C} \to \mathscr{C} \qquad \eta : 1 \to E$$

such that

$$\eta_A : A \to EX$$

is a $\mathscr{W}$-localization for each $A \in \mathscr{C}$. Clearly (E, η) is unique up to natural equivalence, and (E, η) is called the *$\mathscr{W}$-localization functor.*

2.3. *The idempotency of $\mathcal{W}$-localization functors.* Following Adams, I will call a functor and transformation

$$E:\mathcal{C} \to \mathcal{C} \qquad \eta: 1 \to E$$

idempotent if $\eta_{EX} = E\eta_X: EX \to E^2X$ and η_{EX} is an equivalence for all $X \in \mathcal{C}$. It is not hard to show that the $\mathcal{W}$-localization functor (2.2) is idempotent. Conversely, any idempotent functor (E, η) on $\mathcal{C}$ can be obtained as a $\mathcal{W}$-localization where $\mathcal{W}$ consists of all $f: X \to Y \in \mathcal{C}$ such that Ef is an equivalence.

I conclude by recalling from [6, p. 12] a notion which will facilitate the detection of $\mathcal{W}$-local objects.

Definition 2.4. In a category $\mathcal{C}$, a morphism class $\mathcal{W}$ *admits a calculus of left fractions* if:

(i) $\mathcal{W}$ is closed under finite compositions and contains the identities of $\mathcal{C}$.

(ii) Given $X_2 \xleftarrow{w} X_1 \xrightarrow{f} X_3 \in \mathcal{C}$ with $w \in \mathcal{W}$, there exists $X_2 \xrightarrow{g} X_4 \xleftarrow{v} X_3 \in \mathcal{C}$ such that $v \in \mathcal{W}$ and $vf = gw$.

(iii) Given $X_1 \xrightarrow{w} X_2 \overset{f}{\underset{g}{\rightrightarrows}} X_3 \in \mathcal{C}$ with $w \in \mathcal{W}$ and $fw = gw$, there exists $X_3 \xrightarrow{v} X_4 \in \mathcal{C}$ such that $v \in \mathcal{W}$ and $vf = vg$.

It is easy to prove:

LEMMA 2.5. *If $\mathcal{W}$ admits a calculus of left fractions and $D \in \mathcal{C}$, then the following are equivalent:*

(i) *D is $\mathcal{W}$-local.*

(ii) *Each morphism $X \to Y$ in $\mathcal{W}$ induces a surjection* $\mathrm{Hom}(Y, D) \to \mathrm{Hom}(X, D)$.

(iii) *Each morphism $D \to Y$ in $\mathcal{W}$ has a left inverse.*

§3. THE EXISTENCE OF h_*-LOCALIZATIONS

I will state and prove the existence theorem for localizations of spaces with respect to h_*-homology. The proof will rely on the "simplicial homotopy theory with respect to h_*" which I have developed in the Appendix (§10, §11 and §12). First consider:

3.1. *The class of h_*-equivalences.* Let h_* be a generalized homology theory defined on CW pairs and satisfying the limit axiom [1, p. 188]. As usual I will transfer h_* to simplicial pairs (K, L) by letting $h_*(K, L) = h_*(|K|, |L|)$ where "$|\ \ |$" denotes the geometric realization [8, p. 55]. By a slight abuse of notation, let "Ho" denote the pointed homotopy category of Kan complexes or of CW complexes, and let "$\mathbf{h}_*$" denote the class of maps $X \to Y \in \mathrm{Ho}$ inducing isomorphisms $h_* X \approx h_* Y$. This abuse is harmless because the geometric realization provides an equivalence between the Kan and CW pointed homotopy categories [8, pp. 61–66].

The main existence theorem is:

THEOREM 3.2. *Each object of* Ho *has an $\mathbf{h}_*$-localization* (in the sense of 2.1).

I will actually prove a functorial refinement of this theorem involving:

3.3. *The functor* C_{h_*}. Let $\mathscr{S}$ be the category of simplicial sets. By (11.1) there exist a functor and a transformation

$$C_{h_*} : \mathscr{S} \to \mathscr{S} \qquad i : 1 \to C_{h_*}$$

such that:

(i) For each $X \in \mathscr{S}$, $i_X : X \to C_{h_*} X$ is an injection with $h_*(C_{h_*} X, X) = 0$.

(ii) For each $X \in \mathscr{S}$, $C_{h_*} X$ is an *h_*-Kan complex* (12.1), i.e. if $K \subset L$ is a simplicial pair with $h_*(L, K) = 0$, then any map $K \to C_{h_*} X$ can be extended to a map $L \to C_{h_*} X$.

I will prove:

LEMMA 3.4. *For a pointed Kan complex X, the map* $i_X : X \to C_{h_*} X$ *represents the* $\mathbf{h}_*$*-localization of X in* Ho.

This implies (3.2), and also shows that the $\mathbf{h}_*$-localization functor on Ho is induced by the functor C_{h_*} on pointed Kan complexes.

Clearly (3.4) follows from (3.3) and

LEMMA 3.5. *A pointed Kan complex is an h_*-Kan complex if and only if it is* $\mathbf{h}_*$*-local in the pointed homotopy category* Ho.

This is easily proved using (2.5) and the following result observed by Adams.

LEMMA 3.6. *The class* $\mathbf{h}_*$ *admits a calculus of left fractions in* Ho.

Proof. 2.4(i) is obvious. For 2.4(ii), represent w and f by CW inclusions $X_1 \subset X_2$ and $X_1 \subset X_3$, and take $X_4 = X_2 \cup X_3$. For 4.1(iii), represent w by a CW inclusion $X_1 \subset X_2$. Then h_* $(Cyl, Spool) = 0$ where

$$\begin{aligned} Spool &= (0 \times X_2) \cup (I \times X_1/I \times *) \cup (1 \times X_2) \\ Cyl &= I \times X_2/I \times *. \end{aligned}$$

Take v to be represented by the right map of the push-out

$$\begin{tikzcd} Spool \arrow[r] \arrow[d] & X_3 \arrow[d] \\ Cyl \arrow[r] & X_4 \end{tikzcd}$$

where the top map restricts to representatives of f and g on $(0 \times X_2) \cup (1 \times X_2)$.

§4. h_*-LOCALIZATIONS FOR NILPOTENT SPACES

I will use results of [3] and [5] to "compute" the $\mathbf{h}_*$-localizations of nilpotent (e.g. simply connected) spaces, where h_* is any connective homology theory. First recall:

PROPOSITION 4.1 [3, 1.1]. *If h_* is a connective homology theory, then h_* has the same acyclic spaces as $H_*(\ ; A)$, where either $A = Z[J^{-1}]$ or $A = \bigoplus_{p \in J} Z_p$ for some set J of primes.*

Here, $Z[J^{-1}]$ denotes the rationals whose denominators are products of primes in J, and $Z_p = Z/pZ$.

This theorem shows that no new localizations are obtained by using connective homology theories other than the specified $H_*(\ ;A)$.

Now recall from [5, p. 59]:

4.2. *Nilpotent spaces.* A connected object $X \in \mathrm{Ho}$ is *nilpotent* if the group $\Pi_1 X$ is nilpotent and the $\Pi_1 X$-module $\Pi_n X$ is nilpotent for $n \geq 2$ in the following sense. A Π-module is *nilpotent* if it has a finite Π-filtration such that Π acts trivially on the filtration quotients.

For $X \in \mathrm{Ho}$ let $X \to X_A$ denotes the $\mathbf{H}_*(\ ;A)$-localization of X; and let X be connected and nilpotent.

PROPOSITION 4.3. (i) *If* $A = Z[J^{-1}]$, *then* $\Pi_* X_A \approx Z[J^{-1}] \otimes \Pi_* X$, *and* $\tilde{H}_*(X_A; Z) \approx Z[J^{-1}] \otimes \tilde{H}_*(X; Z)$.

(ii) *If* $A = Z_p$, *then there is a splittable short exact sequence*

$$* \to \mathrm{Ext}(Z_{p^\infty}, \Pi_n X) \to \Pi_n X_A \to \mathrm{Hom}(Z_{p^\infty}; \Pi_{n-1} X) \to *.$$

(iii) *If* $A = \bigoplus_{p \in J} Z_p$, *then* $X_A \simeq \prod_{p \in J} X_{Z_p}$.

For a nilpotent group G, $Z[J^{-1}] \otimes G$ denotes the $Z[J^{-1}]$-Malcev completion of G (see [5, p. 128]), while $\mathrm{Ext}(Z_{p^\infty}, G)$ and $\mathrm{Hom}(Z_{p^\infty}, G)$ were defined and studied in [5, pp. 165–182]. For example, if G is finitely generated nilpotent then $\mathrm{Ext}(Z_{p^\infty}, G)$ is p-profinite completion of G and $\mathrm{Hom}(Z_{p^\infty}, G) = *$.

Proof of (4.3) *using* [5]. For a solid ring R (e.g. $R = Z[J^{-1}]$ or $R = Z_p$) and for a pointed connected R-good space X, the R-completion $X \to R_\infty X$ is an $\mathbf{H}_*(\ ;R)$-localization of X by [5, p. 205]. Moreover, a connected nilpotent space is R-good for $R = Z[J^{-1}]$ and $R = Z_p$. Thus 4.3(i) and 4.3(ii) follow from [5, pp. 133, 183]. In 4.3(iii), the product $\prod_{p \in J} X_{Z_p}$ is $\mathbf{H}_*(\ ;A)$-local because its factors are. It now suffices to show $\tilde{H}_*(Y(p); Z_p) = 0$ for each $p \in J$ where $Y(p) = \prod_{q \in J - \{p\}} X_{Z_q}$. This follows from [5, p. 134], because $Y(p)$ is a nilpotent space with uniquely p-divisible homotopy groups.

§5. AN ALGEBRAIC CHARACTERIZATION OF LOCAL SPACES

Throughout this section let $R = Z[J^{-1}]$ or $R = Z_p$. I will show that a connected space $X \in \mathrm{Ho}$ is $\mathbf{H}_*(\ ;R)$-local if and only if the group $\Pi_1 X$ and the $\Pi_1 X$ modules $\Pi_n X$, $n \geq 2$, satisfy certain algebraic conditions. I will need:

5.1. *HR localization theory for groups.* Let $\mathcal{G}$ be the category of groups, and let HR consist of all $\alpha : A \to B \in \mathcal{G}$ such that $\alpha_* : H_i(A; R) \to H_i(B; R)$ is an isomorphism for $i = 1$ and epimorphism for $i = 2$, where A and B act trivially on R. The terminology of §2 now applies, and

THEOREM 5.2. *Every group has an HR-localization.*

A proof of (5.2) and a discussion of this localization are in §7. I will also need:

5.3. *HZ-localization theory for* Π*-modules.* Let Π be a fixed group, let $\mathcal{M}_\Pi$ be the category of left Π-modules, and let HZ consist of all $\alpha : A \to B \in \mathcal{M}_\Pi$ such that $\alpha_* : H_i(\Pi; A) \to H_i(\Pi; B)$ is an isomorphism for $i = 0$ and epimorphism for $i = 1$. The terminology of §2 now applies and,

THEOREM 5.4. *Every* Π*-module has an HZ-localization.*

A proof of (5.4) and a discussion of this localization are in §8.

My algebraic characterization of local spaces is:

THEOREM 5.5. *A connected object* $X \in \mathrm{Ho}$ *is* $\mathbf{H}_*(\ ; R)$-local *if and only if* $\Pi_n X$ *is an HR-local group for* $n \geq 1$ *and* $\Pi_n X$ *is an HZ-local* $\Pi_1 X$*-module for* $n \geq 2$.

This will be proved in §9.

The connectivity condition on X can easily be removed, because an object of Ho is $\mathbf{H}_*$-$(\ ; R)$-local if and only if its components (with arbitrary basepoints) are $\mathbf{H}_*(\ ; R)$-local.

§6. CONSTRUCTIONS OF HOMOLOGY EQUIVALENCES

As a step toward proving the results of §5, I will construct various homology equivalences. I am indebted to E. Dror for the main ideas behind these constructions.

Let $R = Z[J^{-1}]$ or $R = Z_p$ for p prime, let $X \in \mathrm{Ho}$ be connected, let $\alpha : \Pi_1 X \to G$ be a group homomorphism, and let HR be as in (5.1).

LEMMA 6.1. $\alpha \in HR$ *if and only if there exists a map* $f : X \to Y \in \mathrm{Ho}$ *such that* $f_* : H_*(X; R) \approx H_*(Y; R)$ *and* $f_* : \Pi_1 X \to \Pi_1 Y$ *is equivalent to* α.

Proof. The "if" part is clear. For the "only if" part, suppose X is a CW complex. Attach 1-cells and 2 cells to X so as to give $i : X \xrightarrow{\subset} \bar{X}$ with $i_* : \Pi_1 X \to \Pi_1 \bar{X}$ equivalent to α. Then $i_* : H_1(X; R) \approx H_1(\bar{X}; R)$ and there is an obvious commutative diagram

$$\begin{tikzcd}
 & R \otimes \Pi_2 X \arrow[r] \arrow[d] & R \otimes \Pi_2 \bar{X} \arrow[d] & & \\
0 \arrow[r] & H_2(X; R) \arrow[r] \arrow[d] & H_2(\bar{X}; R) \arrow[r] \arrow[d] & H_2(\bar{X}, X; R) \arrow[r] & 0 \\
 & H_2(\Pi_1 X; R) \arrow[r] \arrow[d] & H_2(\Pi_1 \bar{X}; R) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. A diagram chase shows that the composite map $R \otimes \Pi_2 \bar{X} \to H_2(\bar{X}, X; R)$ is onto. Thus there exist elements $\{b_\alpha\}$ in $\Pi_2 \bar{X}$ which go to an R-basis for the free R-module $H_2(\bar{X}, X; R)$. Using attaching maps representing the $\{b_\alpha\}$, attach 3-cells to $\bar{X}$ so as to give $\bar{X} \subset Y$. Then the inclusion $f: X \xrightarrow{\subset} Y$ has the desired properties.

Now let $R = Z[J^{-1}]$, let $X \in \mathrm{Ho}$ be connected, let $\alpha: \Pi_n X \to M$ be a $\Pi_1 X$-module homomorphism for some $n \geq 2$, and let HZ be as in (5.3).

LEMMA 6.2. *$1 \otimes \alpha: R \otimes \Pi_n X \to R \otimes M$ is in HZ if and only if there exists a map $f: X \to Y \in \mathrm{Ho}$ such that $f_*: H_*(X; R) \approx H_*(Y; R)$, $f_*: \Pi_j X \approx \Pi_j Y$ for $j < n$, and $f_*: \Pi_n X \to \Pi_n Y$ is equivalent to α.*

Proof. For the "if" part, suppose $f: X \to Y \in \mathrm{Ho}$ has the specified properties. Then $f_*: H_j(P^n X; R) \to H_j(P^n Y; R)$ is an isomorphism for $j \leq n$ and an epimorphism for $j = n + 1$, where $P^n X$ is the nth Postnikov section of X. By comparing the exact sequence.

(6.3) $H_{n+2}(P^{n-1}X; R) \to H_1(\Pi_1 X; R \otimes \Pi_n X) \to H_{n+1}(P^n X; R) \to H_{n+1}(P^{n-1}X; R) \to H_0(\Pi_1 X; R \otimes \Pi_n X) \to H_n(P^n X; R) \to H_n(P^{n-1}X; R) \to 0$ with the corresponding sequence for Y, it is easy to show $1 \otimes \alpha \in HZ$.

For the "only if" part, suppose X is a CW complex. Attach n-cells and $(n+1)$-cells to X so as to give $i: X \xrightarrow{\subset} \bar{X}$ with $P^{n-1}X \simeq P^{n-1}\bar{X}$ and with $i_*: \Pi_n X \to \Pi_n \bar{X}$ equivalent to α. By (6.3), $i_*: H_j(P^n X; R) \to H_j(P^n \bar{X}; R)$ is an isomorphism for $j \leq n$ and an epimorphism for $j = n + 1$. Thus $i_*: H_n(X; R) \approx H_n(\bar{X}; R)$ and there is an obvious commutative diagram

$$\begin{tikzcd}
 & R \otimes \Pi_{n+1} X \arrow[r] \arrow[d] & R \otimes \Pi_{n+1} \bar{X} \arrow[d] & & \\
0 \arrow[r] & H_{n+1}(X; R) \arrow[r] \arrow[d] & H_{n+1}(\bar{X}; R) \arrow[r] \arrow[d] & H_{n+1}(\bar{X}, X; R) \arrow[r] & 0 \\
 & H_{n+1}(P^n X; R) \arrow[r] \arrow[d] & H_{n+1}(P^n \bar{X}; R) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. A diagram chase shows that the composite map $R \otimes \Pi_{n+1}\bar{X} \to H_{n+1}(\bar{X}, X; R)$ is onto. Thus there exist elements $\{b_\alpha\}$ in $\Pi_{n+1}\bar{X}$ which go to an R-basis for the free R-module $H_{n+1}(\bar{X}, X; R)$. Using attaching maps representing the $\{b_\alpha\}$, attach $(n+2)$-cells to $\bar{X}$ so as to give $\bar{X} \subset Y$. Then the inclusion $f: \bar{X} \xrightarrow{\subset} Y$ has the desired properties.

For $R = Z_p$, (6.2) does not hold and I obtain a less satisfying result. Let $X \in \mathrm{Ho}$ be connected, let $\alpha: \Pi_n X \to M$ be a $\Pi_1 X$-module homomorphism for some $n \geq 2$, and let HZ be as in (5.3). Consider the following conditions:

(6.4) $1 \otimes \alpha: Z_p \otimes \Pi_n X \to Z_p \otimes M$ is in HZ.

(6.5) $\alpha_*: H_0(\Pi_1 X; \mathrm{Tor}(Z_p, \Pi_n X)) \to H_0(\Pi_1 X; \mathrm{Tor}(Z_p, M))$
is onto.

(6.6) There exists a map $f: X \to Y \in \mathrm{Ho}$ such that $f_*: H_*(X; Z_p) \approx H_*(Y; Z_p)$, $f_*: \Pi_j X \approx \Pi_j Y$ for $j < n$, and $f_*: \Pi_n X \to \Pi_n Y$ is equivalent to α.

LEMMA 6.7. (i) *If* (6.4) *and* (6.5), *then* (6.6).

(ii) *If* (6.6), *then* (6.4).

(iii) *If* (6.6), *and* $1 \otimes \alpha: Z_p \otimes \Pi_n X \approx Z_p \otimes M$, *then* (6.5).

The proof is similar to that of (6.2). However, one must use the mod-p Serre spectral sequence for $P^n X \to P^{n-1} X$ instead of (6.3).

§7. HR-LOCALIZATIONS OF GROUPS

Let $R = Z[J^{-1}]$ or $R = Z_p$ for p prime. I will prove the existence Theorem 5.2 for HR-localizations of groups and will give a rather general example.

LEMMA 7.1. *If*

$$\begin{array}{ccc} G_1 & \xrightarrow{t} & G_2 \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\ G_3 & \longrightarrow & G_4 \end{array}$$

is a push-out of groups with $r \in HR$, then $s \in HR$.

Proof. Form a push-out

$$\begin{array}{ccc} K(G_1, 1) & \xrightarrow{h} & K(G_2, 1) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ K(G_3, 1) & \longrightarrow & X \end{array}$$

of pointed connected CW complexes such that f is a cofibration inducing r and h induces t. Then $g_*: \Pi_1 K(G_2, 1) \to \Pi_1 X$ is equivalent to s by Van Kampen's theorem, and clearly $H_i(X, K(G_2, 1); R) = 0$ for $i \leq 2$. This implies $s \in HR$.

LEMMA 7.2. *The class HR admits a calculus of left fractions.*

Proof. 2.4(i) is clear, and 2.4(ii) follows from 7.1. For 2.4(iii), let $G_1 \xrightarrow{w} G_2 \underset{g}{\overset{f}{\rightrightarrows}} G_3 \in \mathcal{G}$ be such that $w \in HR$ and $fw = gw$. Then the "folding" map $\mu: G_2 \coprod_{G_1} G_2 \to G_2$ is in HR because it has an obvious right inverse in HR. Now define $v: G_3 \to G_4$ by the push-out

$$\begin{array}{ccc} G_2 \coprod_{G_1} G_2 & \xrightarrow{(f, g)} & G_3 \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle v} \\ G_2 & \longrightarrow & G_4 \end{array}$$

Clearly $vf = vg$ and $v \in HR$ by (7.1).

Now (2.5), (6.1), and (7.2) easily imply:

LEMMA 7.3. *If* $X \in \mathrm{Ho}$ *is connected and* $X \to D \in \mathrm{Ho}$ *is an* $\mathbf{H}_*(\ ; R)$-*localization, then* $\Pi_1 X \to \Pi_1 D$ *is an HR-localization.*

7.4. *Proof of* 5.1. Since each $K(G, 1) \in \mathrm{Ho}$ has an $\mathbf{H}_*(\ ; R)$-localization by (3.2), each group G has an HR-localization by (7.3).

HR-localizations can be computed for many sorts of groups (e.g. finite, nilpotent, or perfect groups) by using the following result. For a group G let $G = \Gamma_1 G \supset \Gamma_2 G \supset \cdots$ denote the lower central series, and suppose $R \otimes (\Gamma_n G/\Gamma_{n+1} G) = 0$ for some $n \geq 1$. Then:

LEMMA 7.5. *The HR-localization of G is:*

(i) *the obvious map* $G \to Z[J^{-1}] \otimes (G/\Gamma_n G)$ *for* $R = Z[J^{-1}]$, *and*

(ii) *the obvious map* $G \to \mathrm{Ext}(Z_{p^\infty}, G/\Gamma_n G)$ *for* $R = Z_p$.

The reader is referred to §4 and to [5] for an account of the *Malcev completion* $N \to Z[J^{-1}] \otimes N$ and the *Ext-completion* $N \to \mathrm{Ext}(Z_{p^\infty}, N)$ of a nilpotent group N.

Proof. By [13] a short exact sequence of groups $* \to A \to B \to C \to *$ gives an exact sequence

$$H_2(B; R) \to H_2(C; R) \to R \otimes (A/[B, A]) \to H_1(B; R) \to H_1(C; R) \to 0.$$

Thus the quotient map $G \to G/\Gamma_n G$ is in HR. The lemma now follows by (4.3) and (7.3).

Many more examples of HR-local groups can be constructed using the obvious result:

LEMMA 7.6. *The HR-local groups are closed under inverse limits.*

§8. HZ-LOCALIZATIONS OF Π-MODULES

Let Π be a fixed group and let $\mathscr{M}_\Pi$ be the category of left Π-modules. I will prove the existence theorem (5.4) for HZ-localizations in $\mathscr{M}_\Pi$ and will give some general examples.

LEMMA 8.1. *If*

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "r"] & M_2 \arrow[d, "s"] \\ M_3 \arrow[r] & M_4 \end{tikzcd}$$

is a push-out in $\mathscr{M}_\Pi$ *with* $r \in HZ$, *then* $s \in HZ$.

Proof. There is a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "j"] \arrow[dr, "r"] & M_1' \arrow[r] \arrow[d, "r'"] & M_2 \arrow[d, "s"] \\ & M_3 \arrow[r] & M_4 \end{tikzcd}$$

in $\mathcal{M}_\Pi$ such that j is onto and the square is both a pull-back and push-out. The lemma now follows because $r' \in HZ$ and there is a long exact sequence

$$\cdots \to H_1(\Pi; M_4) \to H_0(\Pi; M_1') \to H_0(\Pi; M_2) \oplus H_0(\Pi; M_3) \to H_0(\Pi; M_4) \to 0.$$

LEMMA 8.2. *In $\mathcal{M}_\Pi$, the class HZ admits a calculus of left fractions.*

The proof is similar to that of (7.2).

To prove the existence of HZ-localizations I will need a lemma concerning $H_*(\ ; Z)$-fibrations (10.1). Let $u: X \to Y$ be a Kan fibration of pointed connected Kan complexes such that $u_*: \Pi_1 X \approx \Pi_1 Y$ and $\Pi_i Y = 0$ for $i \geq 2$.

LEMMA 8.3. (i) *If $X \xrightarrow{i} \bar{X} \xrightarrow{v} Y$ is a factorization of u such that $i_*: H_*(X; Z) \approx H_*(\bar{X}; Z)$ and v is an $H_*(\ ; Z)$-fibration, then $v_*: \Pi_1 \bar{X} \approx \Pi_1 Y$.*

(ii) *$u: X \to Y$ is an $H_*(\ ; Z)$-fibration if and only if $\Pi_n X \in \mathcal{M}_{\Pi_1 X}$ is HZ-local for $n \geq 2$.*

Proof of (i). $v_*: \Pi_1 \bar{X} \to \Pi_1 Y \in \mathcal{G}$ is in HZ, because its right inverse $i_*: \Pi_1 X \to \Pi_1 \bar{X} \in \mathcal{G}$ is in HZ by (6.1). Thus by (6.1), v can be factored as $\bar{X} \xrightarrow{r} W \xrightarrow{s} Y$ where r is an injection with $r_*: H_*(\bar{X}; Z) \approx H_*(W; Z)$ and $s_*: \Pi_1 W \approx \Pi_1 Y$. Now r has a left inverse by (10.1), and hence $v_*: \Pi_1 \bar{X} \to \Pi_1 Y$ has a left inverse. Thus $v_*: \Pi_1 \bar{X} \approx \Pi_1 Y$.

Proof of (ii). For the "only if" part it suffices by (2.5) and (8.2) to show for $n \geq 2$ that each map $\Pi_n X \to M \in \mathcal{M}_{\Pi_1 X}$ in HZ has a left inverse. This follows by (6.2) and (10.1). For the "if" part, use (11.1) to factor u as $X \xrightarrow{i} \bar{X} \xrightarrow{v} Y$ such that $i_*: H_*(X; Z) \approx H_*(\bar{X}; Z)$ and v is an $H_*(\ ; Z)$-fibration. Then $i_*: \Pi_1 X \approx \Pi_1 \bar{X}$ by 8.3(i), and $\Pi_n \bar{X} \in \mathcal{M}_{\Pi_1 \bar{X}}$ is HZ-local for $n \geq 2$ by the "only if" part of 8.3(ii). An inductive argument using (6.2) now shows $i_*: \Pi_n X \approx \Pi_n \bar{X}$ for $n \geq 1$, and thus u is homotopy equivalent to v by [8, p. 50]. Hence u is an $H_*(\ ; Z)$-fibration.

8.4. *Proof of* (5.4). For $M \in \mathcal{M}_\Pi$, choose a connected pointed Kan complex X such that $\Pi_1 X = \Pi$ and $\Pi_2 X = M \in \mathcal{M}_\Pi$. By (11.1) the Postnikov map $X \to P^1 X$ can be factored as $X \xrightarrow{i} \bar{X} \xrightarrow{v} P^1 X$ where $i_*: H_*(X; Z) \approx H_*(\bar{X}; Z)$ and v is an $H_*(\ ; Z)$-fibration. Now (6.2) and (8.3) imply that $i_*: \Pi_2 X \to \Pi_2 \bar{X} \in \mathcal{M}_\Pi$ is an HZ-localization.

I will next show that HZ-local modules are closed under various constructions. Clearly:

LEMMA 8.5. *The HZ-local objects of $\mathcal{M}_\Pi$ are closed under inverse limits.*

Less obvious is:

LEMMA 8.6. *If $M_1, M_2 \in \mathcal{M}_\Pi$ are HZ-local and $w: M_1 \to M_2 \in \mathcal{M}_\Pi$, then* coker $(w) \in \mathcal{M}_\Pi$ *is HZ-local.*

Proof. Let

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d, "h"] & X_1 \arrow[d, "f"] \\ X_4 \arrow[r, "g"] & X_2 \end{tikzcd}$$

be a pull-back of pointed connected Kan complexes such that f is a Kan fibration, $\Pi \approx \Pi_1 X_1 \overset{f_*}{\approx} \Pi_1 X_2 \overset{g_*}{\approx} \Pi_1 X_4$, $f_*: \Pi_3 X_1 \to \Pi_3 X_2 \in \mathcal{M}_\Pi$ is equivalent to $w: M_1 \to M_2 \in \mathcal{M}_\Pi$, and all other homotopy groups vanish for X_1, X_2, and X_4. Applying (12.3) and (8.3) to the maps $X_1 \xrightarrow{f} X_2 \to P^1 X_2$, one shows f is an $H_*(\ ;Z)$-fibration. Hence $h: X_3 \to X_4$ is an $H_*(\ ;Z)$-fibration, and $\operatorname{coker}(w) \approx \Pi_2 X_3 \in \mathcal{M}_\Pi$ is HZ-local by (8.3).

Similar methods can be used to prove the following two closure results for HZ-local modules.

LEMMA 8.7. *If M_1, $M_2 \in \mathcal{M}_\Pi$ are HZ-local and $0 \to M_1 \to M_3 \to M_2 \to 0 \in \mathcal{M}_\Pi$ is exact, then $M_3 \in \mathcal{M}_\Pi$ is HZ-local.*

LEMMA 8.8. *If $M \in \mathcal{M}_\Pi$ is HZ-local and $G \to \Pi$ is a group homomorphism, then $M \in \mathcal{M}_G$ is HZ-local.*

I can now construct some examples.

LEMMA 8.9. *If $M \in \mathcal{M}_\Pi$ is nilpotent* (4.2), *then M is HZ-local.*

Proof. Using (8.8) and the homomorphism $\Pi \to * \in \mathcal{G}$, one shows every simple Π-module is HZ-local. The lemma now follows from (8.7).

More generally, for $M \in \mathcal{M}_\Pi$ let

$$M \supset IM \supset I^2 M \supset \cdots$$

be the "lower central series" where $I \subset Z\Pi$ is the augmentation ideal, and suppose $I^n M = I^{n+1} M$ for some $n \geq 0$. Then:

LEMMA 8.10. *The HZ-localization of M is the quotient map $M \to M/I^n M \in \mathcal{M}_\Pi$.*

Proof. A short exact sequence $0 \to A \to B \to C \to 0 \in \mathcal{M}_\Pi$ clearly gives an exact sequence

$$\cdots \to H_1(\Pi; B) \to H_1(\Pi; C) \to A/IA \to H_0(\Pi; B) \to H_0(\Pi; C) \to 0.$$

Thus $M \to M/I^n M$ is in HZ, and the lemma follows from (8.9) since $M/I^n M$ is nilpotent.

Although the HZ-localization functor (2.2) $E: \mathcal{M}_\Pi \to \mathcal{M}_\Pi$ is still somewhat mysterious, one has:

LEMMA 8.11. *$E: \mathcal{M}_\Pi \to \mathcal{M}_\Pi$ is additive and right exact.*

This follows easily because the HZ-local objects of $\mathcal{M}_\Pi$ are closed under products and cokernels.

I conclude with a technical result needed in §9.

LEMMA 8.12 (i) *If $M \in \mathcal{M}_\Pi$ is an $HZ[J^{-1}]$-local group, then so its HZ-localization $EM \in \mathcal{M}_\Pi$.*

(ii) *If $M \in \mathcal{M}_\Pi$ is an HZ-local Π-module, then so is its HZ_p-localization*

$$\operatorname{Ext}(Z_{p^\infty}, M) \in \mathcal{M}_\Pi.$$

Proof. Part (i) follows because E is an additive functor. Part (ii) follows from (8.5), (8.6), and (8.7), using the natural exact sequence

$$0 \to \varprojlim_j{}^1 \operatorname{Hom}(Z_{p^j}, M) \to \operatorname{Ext}(Z_{p^\infty}, M) \to \varprojlim_j \operatorname{Ext}(Z_{p^j}, M) \to 0$$

of [5, p. 166].

§9. PROOF OF THEOREM 5.5

Let $R = Z[J^{-1}]$ or $R = Z_p$. A connected object $X \in \mathrm{Ho}$ will be called *algebraically* $\mathbf{H}_*(\ ; R)$*-local* if $\Pi_n X$ is an HR-local group for $n \geq 1$ and $\Pi_n X$ is an HZ-local $\Pi_1 X$-module for $n \geq 2$. I must prove that $X \in \mathrm{Ho}$ is $\mathbf{H}_*(\ ; R)$-local if and only if it is algebraically $\mathbf{H}_*(\ ; R)$-local.

LEMMA 9.1. *Let X, $Y \in \mathrm{Ho}$ be connected and algebraically $\mathbf{H}_*(\ ; R)$-local. If $f: X \to Y \in \mathrm{Ho}$ induces $f_*: H_*(X; R) \approx H_*(Y; R)$, then f is an equivalence.*

Proof for $R = Z[J^{-1}]$. $f_*: \Pi_1 X \to \Pi_1 Y \in \mathscr{G}$ is an isomorphism because it is in HR by (6.1) and $\Pi_1 X$, $\Pi_1 Y \in \mathscr{G}$ are HR-local. Now $f_*: \Pi_2 X \to \Pi_2 Y \in \mathscr{M}_{\Pi_1 X}$ is an isomorphism because $1 \otimes f_*: R \otimes \Pi_2 X \to R \otimes \Pi_2 Y \in \mathscr{M}_{\Pi_1 X}$ is in HZ by (6.2), $1 \otimes f_*$ is equivalent to $f_*: \Pi_2 X \to \Pi_2 Y \in \mathscr{M}_{\Pi_1 X}$, and $\Pi_2 X, \Pi_2 Y \in \mathscr{M}_{\Pi_1 X}$ are HZ-local. Continuing in this way, one shows $f_*: \Pi_* X \approx \Pi_* Y$.

Proof for $R = Z_p$. As above, $f_*: \Pi_1 X \approx \Pi_1 Y$. Define M, $N \in \mathscr{M}_{\Pi_1 X}$ by the exact sequence $0 \to M \to \Pi_2 X \xrightarrow{f_*} \Pi_2 Y \to N \to 0$. Now $M \in \mathscr{G}$ is HZ_p-local by (7.6), and $N \in \mathscr{G}$ is HZ_p-local because the HZ_p-localization functor, $\mathrm{Ext}(Z_{p^\infty},\ )$, is right exact on abelian groups. Thus the condition $Z_p \otimes M = 0$ (resp. $Z_p \otimes N = 0$) will imply $M = 0$ (resp. $N = 0$) by (7.5). But $1 \otimes f_*: Z_p \otimes \Pi_2 X \approx Z_p \otimes \Pi_2 Y$, because $Z_p \otimes \Pi_2 X, Z_p \otimes \Pi_2 Y \in \mathscr{M}_{\Pi_1 X}$ are HZ-local by (8.6) and $1 \otimes f_*$ is in HZ by (6.7)(ii). Now $N = 0$ becuase $Z_p \otimes N = 0$. Using 6.7(iii) and the exact sequence $\mathrm{Tor}(Z_p, \Pi_2 X) \to \mathrm{Tor}(Z_p, \Pi_2 Y) \to Z_p \otimes M \to 0$, one shows $H_0(\Pi_1 X; Z_p \otimes M) = 0$. Thus $Z_p \otimes M = 0$ by (8.10), because $Z_p \otimes M \in \mathscr{M}_{\Pi_1 X}$ is HZ-local by (8.5) and (8.6). Consequently $M = 0$ and $f_*: \Pi_2 X \approx \Pi_2 Y$. Continuing in this way, one shows $f_*: \Pi_* X \approx \Pi_* Y$.

LEMMA 9.2. *For each connected $X \in \mathrm{Ho}$, there exists a map $f: X \to Y \in \mathrm{Ho}$ such that $f_*: H_*(X; R) \approx H_*(Y; R)$ and Y is algebraically $\mathbf{H}_*(\ ; R)$-local.*

Proof for $R = Z[J^{-1}]$. Using (5.2) and (6.1), construct $f^1: X \to Y^1 \in \mathrm{Ho}$ such that $f_*{}^1: H_*(X; R) \approx H_*(Y^1; R)$ and $f_*{}^1: \Pi_1 X \to \Pi_1 Y^1 \in \mathscr{G}$ is an HR-localization. Using (5.4) and (6.2), construct $f^2: Y^1 \to Y^2 \in \mathrm{Ho}$ such that $f_*{}^2: H_*(Y^1; R) \approx H_*(Y^2; R), f_*{}^2: \Pi_1 Y^1 \approx \Pi_1 Y^2$, and $f_*{}^2: \Pi_2 Y^1 \to \Pi_2 Y^2 \in \mathscr{M}_{\Pi_1 Y^1}$ is equivalent to the obvious composition $\Pi_2 Y^1 \to R \otimes \Pi_2 Y^1 \to E(R \otimes \Pi_2 Y^1)$ where E is the HZ-localization functor. Then $\Pi_1 Y^2, \Pi_2 Y^2 \in \mathscr{G}$ are HR-local by (8.12) and $\Pi_2 Y^2 \in \mathscr{M}_{\Pi_1 Y^2}$ is HZ-local. Continuing in this way, one obtains a sequence $X \to Y^1 \to Y^2 \to Y^3 \to \cdots$ from which the desired map $X \to Y$ can be constructed by means of an infinite mapping cylinder.

Proof for $R = Z_p$. Construct $f^1: X \to Y^1$ as above. Using (5.4), (6.2), and (6.7(i)) construct $f^2: Y^1 \to Y^2 \in \mathrm{Ho}$ such that $f_*{}^2: H_2(Y^1; Z_p) \approx H_2(Y^2; Z_p), f_*{}^2: \Pi_1 Y^1 \approx \Pi_1 Y^2$, and $f_*{}^2: \Pi_2 Y^1 \to \Pi_2 Y^2 \in \mathscr{M}_{\Pi_1 Y^1}$ is equivalent to the obvious composition $\Pi_2 Y^1 \to E(\Pi_2 Y^1) \to \mathrm{Ext}(Z_{p^\infty}, E(\Pi_2 Y^1))$ where E is the HZ-localization functor. The above use of 6.7(i) is justified because the map $A \to \mathrm{Ext}(Z_{p^\infty}, A)$ induces an isomorphism $Z_p \otimes A \to Z_p \otimes \mathrm{Ext}(Z_{p^\infty}, A)$ and an epimorphism $\mathrm{Tor}(Z_p, A) \to \mathrm{Tor}(Z_p, \mathrm{Ext}(Z_{p^\infty}, A))$ for any abelian group A. Now $\Pi_1 Y^2, \Pi_2 Y^2 \in \mathscr{G}$ are HZ_p-local by (7.5) and $\Pi_2 Y^2 \in \mathscr{M}_{\Pi_1 Y^1}$ is HZ-local by (8.12). The proof is completed as before.

9.3. *Proof of* 5.5. For the "if" part, suppose $X \in \mathrm{Ho}$ is connected and algebraically $\mathbf{H}_*(\ ; R)$-local. To prove X is $\mathbf{H}_*(\ ; R)$-local, it suffices by (2.5) and (3.6) to prove that each map $X \to Y \in \mathrm{Ho}$ in $\mathbf{H}_*(\ ; R)$ has a left inverse. This can be obtained by first using (9.2) to construct $Y \to W \in \mathrm{Ho}$ in $\mathbf{H}_*(\ ; R)$ with W algebraically $\mathbf{H}_*(\ ; R)$-local, and then using (9.2) to show that the composition $X \to Y \to W \in \mathrm{Ho}$ is an equivalence. The "only if" part now follows from the "if" part and (9.2).

Remark 9.4. Our proof of Lemma 9.2 can now be regarded as a step by step construction of the $\mathbf{H}_*(\ ; R)$-localization.

APPENDIX

In this Appendix I will develop a version of simplicial homotopy theory in which the h_*-homology equivalences play the role of weak homotopy equivalences, where h_* is a generalized homology theory as in (3.1). This simplicial theory with respect to h_* (like ordinary simplicial theory) fits very nicely in Quillen's "homotopical algebra" framework [10], [11], and I will so present it.

§10. SIMPLICIAL HOMOTOPY THEORY MODULO h_*

Let $\mathcal{S}$ denote the category of simplicial sets (see [5], [8]).

10.1. *Definitions.* A map $f: K \to L \in \mathcal{S}$ is a *weak h_*-equivalence* if $f_*: h_*(K) \approx h_*(L)$. A map in $\mathcal{S}$ is an *h_*-cofibration* if it is a cofibration (i.e. injection) in $\mathcal{S}$. A map $u: X \to Y \in \mathcal{S}$ is an *h_*-fibration* if it has the right lifting property with respect to each map $i: A \to B \in \mathcal{S}$ which is a weak h_*-equivalence and h_*-cofibration, i.e. for each commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "u"] \\ B \arrow[r] \arrow[ur, dashed, "e"] & Y \end{tikzcd}$$

there exists a map e making the triangles commute. Clearly, any h_*-fibration is a Kan fibration.

I will show that the above notions satisfy Quillen's axioms for a closed model category [11, p. 233]. This will lay the foundation for a Quillen-like homotopy theory. Indeed Quillen has shown [10] that any closed model category (or its associated pointed category) gives rise to much of the familiar homotopy machinery, e.g. the homotopy relations for maps, loops and suspensions, fibration and cofibration exact sequences, Toda brackets, etc.

THEOREM 10.2. *The notions of* (10.1) *in the category* $\mathcal{S}$ *satisfy Quillen's closed model category axioms:*

*CM*1. $\mathcal{S}$ is closed under finite direct and inverse limits.

*CM*2. If f and g are maps such that gf is defined, then if two of f, g, and gf are weak h_*-equivalences, so is the third.

*CM*3. If f is a retract of g and g is a weak h_*-equivalence, an h_*-fibration, or an h_*-cofibration, then so is f.

*CM*4. Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "u"] \\ B \arrow[r] \arrow[ur, dashed, "e"] & Y \end{tikzcd}$$

where i is an h_*-cofibration, u is an h_*-fibration, and either i or u is a weak h_*-equivalence, then there exists a map e making the triangles commute.

*CM*5. Any map f can be factored in two ways:

(i) $f = ui$, where i is an h_*-cofibration and u is an h_*-fibration which is a weak h_*-equivalence.

(ii) $f = ui$, where u is an h_*-fibration and i is an h_*-cofibration which is a weak h_*-equivalence.

Proof. Using the closed model category axioms for ordinary weak equivalences, cofibrations, and Kan fibrations in $\mathcal{S}$, it is straightforward to show that a map $u: X \to Y \in \mathcal{S}$ is an h_*-fibration and weak h_*-equivalence if and only if u is a Kan fibration and weak equivalence. It is now easy to deduce all of the axioms except CM5(ii), which follows from the main result (11.1) of the next section.

§11. A FACTORIZATION THEOREM

This section is devoted to proving the following key theorem which was used in §10 and elsewhere.

THEOREM 11.1. *For each map $f: X \to Y \in \mathcal{S}$ there is a natural factorization $X \xrightarrow{i} E_f \xrightarrow{u} Y$ such that u is an h_*-fibration and i is an h_*-cofibration which is a weak h_*-equivalence.*

Let c be a fixed infinite cardinal number which is at least equal to the cardinality of $h_*(pt)$. For $A \in \mathcal{S}$ let $\# A$ denote the number of non-degenerate simplices in A. We shall implicitly use the easily proved fact that $h_*(A, B)$ has at most c elements if $\# A \leq c$.

LEMMA 11.2. *If (K, L) is a simplicial pair with $h_*(K, L) = 0$, then there exists a subcomplex $A \subset K$ such that $\# A \leq c$, $A \not\subset L$, and $h_*(A, A \cap L) = 0$.*

Proof. The desired A is given by the union $A = \bigcup_{n \geq 1} A_n$ where

$$A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$$

is a sequence subcomplexes of K such that $\# A_n \leq c$, $A_n \not\subset L$, and the map

$$h_*(A_n, A_n \cap L) \to h_*(A_{n+1}, A_{n+1} \cap L)$$

is zero for each $n \geq 1$. To inductively construct $\{A_n\}$, first choose $A_1 \subset K$ such that $\# A_1 \leq c$ and $A_1 \not\subset L$. Then, given A_n, choose for each element $x \in h_*(A_n, A_n \cap L)$ a finite complex $F_x \subset K$ such that x goes to zero in $h_*(A_n \cup F_x, (A_n \cup F_x) \cap L)$. This is possible since $h_*(K, L) = 0$ and h_* satisfies the limit axiom. Finally, let A_{n+1} be the union of A_n with all F_x for $x \in h_*(A_n, A_n \cap L)$.

LEMMA 11.3. *Let $u : X \to Y \in \mathcal{S}$ be a map which has the right lifting property with respect to each inclusion map $A \xrightarrow{\subset} B \in \mathcal{S}$ such that $h_*(B, A) = 0$ and $\# B \leq c$. Then u is an h_*-fibration.*

Proof. If suffices to show that u has the right lifting property with respect to each inclusion map $L \to K \in \mathcal{S}$ such that $h_*(K, L) = 0$. This follows by transfinite induction because, for each such pair (K, L), there exists $M \in \mathcal{S}$ such that $L \subset M \subset K$, $L \neq M$, $h_*(M, L) = 0$, and u has the right lifting property with respect to $L \xrightarrow{\subset} M$. Indeed, one can choose M to be $A \cup L$ where A is as in (11.2).

LEMMA 11.4. *For each map $f: X \to Y \in \mathcal{S}$ there is a natural factorization $X \xrightarrow{j} F_f \xrightarrow{v} Y$ such that:*

(i) *j is an injection with $h_*(F_f, X) = 0$, and*

(ii) *for each inclusion $i : A \xrightarrow{\subset} B \in \mathcal{S}$ with $h_*(B, A) = 0$ and $\# B \leq c$, and for each commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[r, "j"] & F_f \arrow[d, "v"] \\
B \arrow[rru, dashed, "e"] \arrow[rr] & & Y
\end{tikzcd}$$

there exists a map e such that the triangles commute.

Proof. Choose a set $\{i_\alpha : A_\alpha \xrightarrow{\subset} B_\alpha\}_{\alpha \in I}$ of inclusion maps in $\mathcal{S}$ with $h_*(B_\alpha, A_\alpha) = 0$ and $\# B_\alpha \leq c$, and such that each inclusion map with these properties is isomorphic to some i_α. For each $\alpha \in I$, let S_α be the set of maps from i_α to f. Using the obvious commutative diagram

$$\begin{tikzcd}
\bigcup\limits_{s \in S_\alpha} \bigcup\limits_{\alpha \in I} A_\alpha \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\bigcup\limits_{s \in S_\alpha} \bigcup\limits_{\alpha \in I} B_\alpha \arrow[r] & Y
\end{tikzcd}$$

where "$\bigcup$" denotes the disjoint union, define F_f as the push-out of the top and left maps, define $j : X \to F_f$ as the induced cofibration of the left map, and define $v : F_f \to Y$ by the universal property of push-outs.

11.5. *Proof of* 11.1. Let S be the section of the first ordinal of cardinality greater than c. Using transfinite induction, define a commutative diagram in $\mathcal{S}$

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "f"] & X_0 \arrow[r, "i_0"] \arrow[d, "u_0"] & X_1 \arrow[r] \arrow[d, "u_1"] & \cdots \arrow[r] & X_s \arrow[r, "i_s"] \arrow[d, "u_s"] & X_{s+1} \arrow[r] \arrow[d, "u_{s+1}"] & \cdots \\ Y \arrow[r, "1"] & Y \arrow[r, "1"] & Y \arrow[r] & \cdots \arrow[r] & Y \arrow[r, "1"] & Y \arrow[r] & \cdots \end{tikzcd}$$

for $s \in S$ as follows. The map $X_0 \xrightarrow{u_0} Y$ equals $X \xrightarrow{f} Y$; the factorization $X_s \xrightarrow{i_s} X_{s+1} \xrightarrow{u_{s+1}} Y$ equals the factorization $X_s \xrightarrow{j} F_{u_s} \xrightarrow{v} Y$ of (11.4); and if s is a limit ordinal, then $X_s = \varinjlim_{n<s} X_n$ and $u_s = \varinjlim_{n<s} u_n$. To obtain the desired factorization $X \xrightarrow{i} E_f \xrightarrow{u} Y$, let $E_f = \varinjlim_{s \in S} X_s$, let $u = \varinjlim_{s \in S} u_s$, and let i be the obvious injection. In order to show that u is an h_*-fibration, it suffices by 11.3 to show that it has the right lifting property with respect to each inclusion map $A \xrightarrow{\subset} B \in \mathscr{S}$ such that $h_*(B, A) = 0$ and $\# B \leq c$. This property follows easily from 11.4(ii), because the image of each map $A \to E_f$ will be contained in X_s for some $s \in S$.

§12. HOMOTOPY INVERSE LIMITS OF h_*-KAN COMPLEXES

Definition 12.1. A simplicial set X is an *h_*-Kan complex* if $X \to *$ is an h_*-fibration.

I will show that the h_*-Kan complexes are closed under all sorts of homotopy inverse limits. This is of interest because the pointed h_*-Kan complexes represent the $\mathbf{h}_*$-local homotopy types.

For $A, X \in \mathscr{S}$ let $\hom(A, X) \in \mathscr{S}$ denote the simplicial function complex [8, p. 17]; and for

$$i : A \to B \in \mathscr{S} \qquad u : X \to Y \in \mathscr{S}$$

let

$$(i^*, u_*) : \hom(B, X) \to \hom(i, u) \in \mathscr{S}$$

denote the obvious simplicial map

$$\hom(B, X) \to \hom(A, X) \times_{\hom(A, Y)} \hom(B, Y).$$

PROPOSITION 12.2. *Let $i : A \to B \in \mathscr{S}$ be an h_*-cofibration and let $u : X \to Y \in \mathscr{S}$ be an h_*-fibration. Then (i^*, u_*) is an h_*-fibration which is a weak h_*-equivalence if either i or u is a weak h_*-equivalence.*

Proof. The required right lifting properties for (i^*, u_*) can be deduced from the "adjoint" of (12.2). Namely, if $j : K \to L \in \mathscr{S}$ is an h_*-cofibration, then

$$(K \times B) \textstyle\bigcup_{(K \times A)} (L \times A) \to L \times B$$

is an h_*-cofibration, which is a weak h_*-equivalence if either i or j is such.

Note. In using (12.2) it is useful to recall that a map in $\mathscr{S}$ is a Kan fibration and weak equivalence if and only if it is an h_*-Kan fibration and weak h_*-equivalence.

PROPOSITION 12.3. *For $X \xrightarrow{u} Y \xrightarrow{v} W \in \mathscr{S}$, suppose v and vu are h_*-fibrations. If u is a Kan fibration, then u is an h_*-fibration.*

COROLLARY 12.4. *If* $u: X \to Y \in \mathscr{S}$ *is a Kan fibration of* h_**-Kan complexes, then* u *is an* h_**-fibration.*

Proof of 12.3. Letting $i: A \to B \in \mathscr{S}$ be an h_*-cofibration which is a weak h_*-equivalence, it will suffice to show that (i^*, u_*) is surjective in dimension 0. For this it suffices to show (i^*, u_*) is a weak equivalence, because it is a Kan fibration by the usual "non-h_*" version of (12.2). But (i^*, v_*) and $(i^*, (vu)_*)$ are weak equivalences by (12.2), and $(i^*, (vu)_*)$ factors as

$$\hom(B, X) \xrightarrow{(i^*, u_*)} \hom(i, u) \longrightarrow \hom(i, vu)$$

where the second map is an induced fibration of (i^*, v_*). Thus (i^*, u_*) is a weak equivalence.

It is now easy to show that h_*-Kan complexes are closed under familiar sorts of homotopy inverse limits.

PROPOSITION 12.5. *If* $\{X_\alpha\}$ *are* h_**-Kan complexes, then so is* ΠX_α.

PROPOSITION 12.6. *If* X *is an* h_**-Kan complex and* $K \in \mathscr{S}$, *then* $\hom(K, X)$ *is an* h_**-Kan complex.*

PROPOSITION 12.7. *Let* X, Y, $B \in \mathscr{S}$ *be* h_**-Kan complexes, and let*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X \arrow[d, "u"] \\ Y \arrow[r] & B \end{tikzcd}$$

be a pull-back with u *a Kan fibration. Then* E *is an* h_**-Kan complex.*

PROPOSITION 12.8. *If* $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ *is a tower of Kan fibrations with each* X_n *an* h_**-Kan complex, then* $\varprojlim X_n$ *is an* h_**-Kan complex.*

In [5, p. 295] we defined the *homotopy inverse limit*, $\operatorname{ho}\varprojlim X \in \mathscr{S}$, for an arbitrary small diagram X of simplicial sets; and we showed that $\operatorname{ho}\varprojlim X$ had the "right" homotopy type for familiar diagrams of Kan complexes. Thus the following theorem generalizes the above propositions.

THEOREM 12.9. *If* X *is a small diagram of* h_**-Kan complexes, then* $\operatorname{ho}\varprojlim X$ *is an* h_**-Kan complex.*

In view of [5, p. 303], this theorem follows from:

PROPOSITION 12.10. *If* $\mathbf{X}$ *is a fibrant cosimplicial simplicial set such that* $\mathbf{X}^n$ *is an* h_**-Kan complex for* $n \geq 0$, *then* $\operatorname{Tot} \mathbf{X}$ *is an* h_**-Kan complex.*

Proof. I will freely use the notation and results of [5, Ch. X]. Using (12.6) and the fibre squares (see [5, p. 287])

$$\begin{tikzcd} M^n_{k+1}\mathbf{X} \arrow[r] \arrow[d] & \mathbf{X}^n \arrow[d] \\ M^n_k\mathbf{X} \arrow[r] & M^{n-1}_k\mathbf{X} \end{tikzcd}$$

it is not hard to show that each $M_k^n \mathbf{X}$ is an h_*-Kan complex. Thus the natural maps (see [5, p. 274]) $s : \mathbf{X}^{n+1} \to M_n^n \mathbf{X}$ are h_*-Kan fibrations by (12.4). Since there are pull-backs

$$\begin{tikzcd} \mathrm{Tot}_{n+1}\mathbf{X} \arrow[r] \arrow[d] & \hom(\Delta[n+1], \mathbf{X}^{n+1}) \arrow[d, "(i^*, s_*)"] \\ \mathrm{Tot}_n \mathbf{X} \arrow[r] & \hom(i, s) \end{tikzcd}$$

where $\Delta[n+1]$ is the standard $(n+1)$ simplex and $i : \dot{\Delta}[n+1] \to \Delta[n+1]$ is the inclusion of its n-skeleton, the maps $\mathrm{Tot}_{n+1}\mathbf{X} \to \mathrm{Tot}_n \mathbf{X}$ are h_*-fibrations by (12.2). The lemma now follows from (12.8).

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